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Alternation with a Null Point

CHARLES B. DUNHAM

Computer Science Department, University of Western Ontario, London, Canada

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In some approximation problems on an interval, all functions of interest vanish at one endpoint. One classical case is where all functions vanish at zero. Another is where all functions decay to zero at $+\infty$. A necessary and sufficient condition for the best approximation is given in terms of alternation for the case where zero is the null point. The results can be extended to any other interval by a 1 : 1 change of variable, and to functions taking any other fixed value at an endpoint (we subtract the fixed value from such functions to convert the problem into standard form).

Let $CZ[0, \alpha]$ be the space of continuous functions on $[0, \alpha]$ with a zero at the point 0. For $g \in CZ[0, \alpha]$ define

$$\| g \| = \sup\{| g(x)| : 0 \leqslant x \leqslant \alpha\}.$$

Let F be an approximating function with parameter space P such that $F(A, \ ) \in CZ[0, \alpha]$ for all $A \in P$. The Chebyshev approximation problem is given $f \in CZ[0, \alpha]$ to find $A^* \in P$ to minimize $\| f - F(A, \ )\|$. Such a parameter A^* is called best and $F(A^*, \ )$ is called a best approximation to f.

We give necessary and sufficient conditions on (F, P) for best approximations to be characterized by alternation. We follow the approach of Rice [5, pp. 17–21].

DEFINITION. $g \in CZ[0, \alpha]$ has n alternations if there exists $\{x_0, ..., x_n\}$, $0 \leqslant x_0 < \cdots < x_n \leqslant \alpha$ such that

$$| g(x_0)| = \| g \|; g(x_i) = (-1)^i g(x_0) \qquad i = 1, ..., n.$$

DEFINITION. F has property Z of degree n at A if $F(A, \ ) - F(B, \ )$ having n zeros on $(0, \alpha]$ implies that $F(A, \ ) \equiv F(B, \ )$.

DEFINITION. F has property a of degree n at A if given

(i) an integer m, $0 \leqslant m < n$,

(ii) a set $\{x_1, ..., x_m\}$ with $0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = \alpha$,

(iii) ϵ with $0 < \epsilon < \min\{x_{j+1} - x_j : j = 0,..., m\}/2$, and

(iv) a sign σ,

there exists $B \in P$ with $\| F(A, \ ) - F(B, \ )\| < \epsilon$ and

$$\begin{aligned} \operatorname{sign}(F(B, x) - F(A, x)) &= \sigma & \epsilon < x \leqslant x_1 - \epsilon \\ &= \sigma(-1)^i & x_i + \epsilon \leqslant x \leqslant x_{i+1} - \epsilon \\ &= \sigma(-1)^m & x_m + \epsilon \leqslant x \leqslant \alpha. \end{aligned}$$

In the case $m = 0$, we require

$$\operatorname{sign}(F(B, x) - F(A, x)) = \sigma \qquad \epsilon < x \leqslant \alpha.$$

DEFINITION. F has degree n at A if F has property Z and property a of degree n at A.

DEFINITION. A zero x of a continuous function g on $[0, \alpha]$ is called a double zero if x is an interior point of $[0, \alpha]$ and g does not change sign at x.

LEMMA 1. *Let f have degree n at A and degree m at B, $n \leqslant m$. If $F(A, \ ) - F(B, \ )$ has n zeros on $(0, \alpha)$, counting double zeros twice, then $F(A, \ ) \equiv F(B, \ )$.*

Proof of the lemma involves showing that if $F(A, \ ) - F(B, \ )$ has n zeros, counting double zeros twice, there is $C \in P$ with $F(A, \ ) - F(C,)$ having n simple zeros on $(0, \alpha]$. The arguments of [3, p. 299] can be adapted to show this. Alternatively we can apply [2, Lemma 7].

We develop a result related to a result of de la Vallée-Poussin [4, p. 60].

LEMMA 2. *Let F have a degree at all parameters in P. Let F have degree n at A. Let $f - F(A, \ )$ alternate in sign on the sequence $\{x_0 ,..., x_n\}$, $0 < x_0 < \cdots < x_n \leqslant \alpha$. Then for $B \in P$ with $F(B, \ ) \not\equiv F(A, \ )$,*

$$\min\{| f(x_i) - F(A, x_i)| : i = 0,..., n\} < \max\{| f(x_i) - F(B, x_i)| : i = 0,..., n\}.$$

Proof. Suppose that the inequality fails for some $B \in P$. Assume without loss of generality that $f(x_0) - F(A, x_0) > 0$. We have

$$\begin{aligned} F(A, x_0) &\leqslant F(B, x_0), \\ F(A, x_1) &\geqslant F(B, x_1), \\ F(A, x_2) &\leqslant F(B, x_2),..., \end{aligned}$$

and it is seen that $F(A, \ ) - F(B, \ )$ must have n zeros, counting double zeros twice, on $[x_0 , x_n]$. But by Lemma 1, this can only happen if $F(A, \ ) \equiv F(B, \ )$.

Two consequences of the lemma are that n alternations of $f - F(A, \ )$ are sufficient for A to be best to f and that if this occurs, $F(A, \ )$ is a unique best approximation. n alternations of $f - F(A, \ )$ are also necessary for A to be best. Suppose $f - F(A, \ )$ does not alternate n times on $[0, \alpha]$.

Let

$$\sigma = \inf\{x : |f(x) - F(A, x)| \geqslant \|f - F(A, \ )\|/2\}.$$

Then $f - F(A, \ )$ does not alternate n times on $[\sigma, \alpha]$. By [5, Lemma 7-5, pp. 18–19], $F(A, \ )$ is not best to f on $[\sigma, \alpha]$ and there exists better B with $\| F(A, \ ) - F(B, \ )\| < \|f - F(A, \ )\|/2$; hence, such B are also better on $[0, \alpha]$. We have

THEOREM. *Let F have a degree at all parameters in P. Let the degree of F at A be n. A necessary and sufficient condition that $F(A, \ )$ be best to f is that $f - F(A, \ )$ alternate n times. Best approximations are unique.*

We have established a sufficient condition for an alternating theory that (F, P) have a degree. We now show that this is necessary.

DEFINITION. F has property NS of degree n at A if a necessary and sufficient condition for $F(A, \ )$ to be best to f is that $f - F(A, \ )$ alternate n times.

LEMMA 3. *If F has property NS of degree n at A then F has property Z of degree n at A.*

Proof. Suppose $F(A, \ ) - F(B, \ )$ has n zeros on $(0, \alpha]$. Then there is $\sigma > 0$ such that it has n zeros on $(\sigma, \alpha]$. We use the proof of [5, Lemma 7-7, pp. 19–21] and extend f_1, f_2 to $[0, \sigma)$ such that $f_1 - F(A, \ ), f_1 - F(B, \ ), f_2 - F(A, \ ), f_2 - F(B, \ )$ do not attain their norm on $[0, \sigma)$.

LEMMA 4. *If F has property NS of degree n at A, then F has property of degree n at A.*

We use a proof almost identical to that of [5, Lemma 7-8, p. 21].

THEOREM. *A necessary and sufficient condition that (F, P) have property NS is that (F, P) have a degree.*

The theory of [6] for ordinary alternation can easily be extended to cover the extended theory of alternation of this paper.

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